Dancing Coin

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Abstract

This article is on International Young Physicist's Tournament (IYPT) 2018 (see http://iypt.org/Home) problem number three, *Dancing Coin*. The task statement as issued by the IYPT-commission reads as follows: *Take a strongly cooled bottle and put a coin on its neck. Over time you will hear a noise and see movements of the coin. Explain this phenomenon and investigate how the relevant parameters affect the dance.* A theoretical model is presented by which the heat flows into the system can be modelled in order to predict the change in pressure in the bottle. This allows to predict the most important entity involved in the phenomenon, namely the critical pressure at which a lift-off can occur, as well as other dynamic processes involved. In a next step, a possible explanation for the occurrence of the sound is given. Finally, the model quality is evaluated in the light of measured data.
The determining parameter when considering the movement of the coin that has been found during research is coin mass as it affects the pressure dynamics of the system the most. The sound phenomenon, on the other hand, can not be explained to full satisfaction as a result of the resonance of the bottle and the damping of natural frequencies of the coin.

1 Qualitative description

We consider the bottle - i.e. the bottle wall as well as the air that is sealed within it through the coin on top of the bottle neck - to be our system of concern. After cooling, a little water is added to the top of the bottle in order to seal the system when the coin is put on the bottle neck (without the water, no sealing of the system could be observed). At this point, the temperature within the bottle T_i is significantly lower than the ambient temperature T_a. At time $t = 0$, which is considered to be the moment the system is sealed, the pressure p within the system is equivalent to the ambient pressure and is depicted as p_0. There is a number of n air particles in the system. The temperature gradient[1] $\vec{\nabla}T$ created by the temperature difference between T_i and T_a directs the flow of heat into the system according to *Fourier's law of heat transfer*[2]. Thus, an increase in temperature of $+\Delta T$ occurs within the system. This increases the pressure by $+\Delta p$. As heat continues to flow into the system (at a slower rate with decreasing temperature difference), eventually a critical temperature increase $+\Delta T_{crit}$ and along with it a critical pressure increase $+\Delta p_{crit}$ is reached. At this point, the pressure within the bottle is enough to lift the coin, temporarily unsealing the system. This *lift-off* results in a sudden drop in pressure of precisely the critical pressure difference in a proper lift-off back to ambient pressure as well as a decrease of $-\Delta n$ air particles in the system due to the pressure adjustment of the system. This means that over time, there will be a net loss of air particles in the system, which makes sense as the cool air is denser at time $t = 0$. As the lift-off terminates, the coin comes crashing back onto the bottle neck and a sound can be perceived. In perfect experimental conditions, the system seals itself again as soon as the coin is back on the bottle neck, and the temperature in the system, which increases continuously with time, creates again a pressure difference that builds up to the critical temperature difference, eventually resulting in another lift-off. This process repeats until the pressure difference that can be reached is lower than the critical temperature difference, which depends on the difference in temperatures between the system and the (constant) ambient temperature.

2 Theoretical model

2.1 Temperature evolution

First consider the flow of heat into the system. A good approximation is given by Newton's law of cooling, which says that the rate of change of temperature[3] $\dot{T}$ is proportional to the temperature difference ΔT, i.e.

[1] Column vector whose entries are the partial derivatives of the corresponding function with respect to the space directions x, y and z, i.e. $\vec{\nabla}T = \left(\frac{\partial T}{\partial x}\ \frac{\partial T}{\partial y}\ \frac{\partial T}{\partial z}\right)^T$

[2] Heat flux $\vec{q}$ ($[q] = \frac{J}{m^2}$) is equivalent to $-\lambda\vec{\nabla}T$ where λ is the thermal conductivity, a tensor material property

[3] $\dot{T} = \frac{\mathrm{d}T}{\mathrm{d}t}$

$$\dot{T} = -k\left(T(t) - T_a\right) \tag{2.1}$$

The general solution to this first-order differential equation is $T(t) = \pm A \cdot e^{-kt} + T_a$, where, in our case, $A = |T_a - T_i|$. The idea is to predict the warming coefficient k that is dependant on material properties. To do this, we use the notion that the heat flow rate $\dot{Q} = I$ is given by

$$\dot{Q} = I = \Delta T \sum_{i=1}^{n} U_i \cdot S_i \tag{2.2}$$

where U is the thermal transmittance and S is the interaction surface area. We can see from this that we consider several parallel heat flows into the system - one through the coin and one through the bottle wall. The corresponding heat flows have different thermal resistances Ψ that are given through material properties and states of convection and other external conditions. Note that thermal transmittance is Ψ^{-1}. In addition to equation (2.2), we need the following equation:

$$\dot{Q} = \dot{T} \cdot \sum_{i=1}^{n} c_i \cdot mi \tag{2.3}$$

which allows the computation of k according to

$$k = \frac{\sum_{i=1}^{n} U_i \cdot S_i}{\sum_{i=1}^{n} c_i \cdot m_i} \tag{2.4}$$

where

$$\sum_{i=1}^{n} U_i \cdot S_i = \sum_{i=1}^{n} \frac{S_i}{\Psi_i} = \left(\frac{A_b}{\frac{1}{\alpha_{eb}} + \frac{d_b}{\lambda_b} + \frac{1}{\alpha_{bi}}} + \frac{A_c}{\frac{1}{\alpha_{ec}} + \frac{d_c}{\lambda_c} + \frac{1}{\alpha_{ci}}} \right) \tag{2.5}$$

where A_b is the bottle surface, A_c is the coin surface, α_{eb} is the heat transfer coefficient from exterior air to the bottle (dependant on state of convection), d_b the thickness of the bottle wall, λ_b the thermal conductivity of the bottle wall material and α_{bi} the heat transfer coefficient from bottle wall to the interior air of the system (Kammer and Mgeladze, 2014). Similarly, the denominator in equation (2.5) is composed of the same coefficients for the coin and therefore with different values. As there are no precise values of these coefficients in literature, and measuring them would require advanced experimental skill, the warming coefficient k was fitted to data of temperature evolution. It can, however, be concluded from equation (2.5) what the determining factors in the rate of change of temperature in the system must be. Note that we do not consider radiation, because at a temperature difference of $40K$, the absorbed radiation power is less than one percent of the absorbed conductive power, which makes radiation negligible.

2.2 Equation of state for lift-offs

Treating the air in the system as being an ideal gas, it is possible to formulate an equation of state when a lift-off occurs. The ideal gas law is $pV = nRT$, where p is pressure, V is volume, n is number of particles, R is the ideal gas constant and T is the absolute temperature. Rewriting this equation, we obtain

$$\frac{am_c + 4\pi r\sigma}{A'_c} = \frac{\rho}{M} R\Delta T_{crit} \tag{2.6}$$

where the left side of the equation depicts the force required to lift the coin, which is precisely a pressure factor[4] a times the coin mass m_c, to which the surface tension force[5] $4\pi r\sigma$ has to be added and the sum of the two divided by the coin-system interaction area A'_c. This is equivalent to the right side where ρM^{-1} is number of molecules divided by bottle volume (number of molecules can be expressed as density ρ times volume V divided by molar mass M, but V occurs on both sides of the equation, it cancels out) times the ideal gas constant R times the critical temperature difference required for a lift-off ΔT_{crit}. From the equation of state (equation (2.6)), the pressure difference required for a lift off Δp in dependency of the coin mass m can be derived according to

$$\Delta p = \frac{am_c g + 4\pi r\sigma}{A_c} \tag{2.7}$$

2.3 Discrete lift-off model

We now seek to predict the evolution of the time it takes between two subsequent lift-offs. For this, we must consider the following discrete equations. First consider how air density develops between two lift-offs. The initial air density in the bottle is given by the coefficient of volume expansion γ and the initial absolute temperature difference ΔT_i (i.e. the difference between the onset temperature of the measurement and the temperature pertaining to ρ_i).

[4] This factor a is needed due to possible pressure inhomogeneities in the system that will cause the force enacted be $p\vec{A'_c}$, where $\vec{A'_c}$ is the normal vector of the interaction area between the air in the system and the surface of the coin, to affect the coin off its pivot. Therefore, factor a is introduced with $a \in (0.5; 1)$

[5] σ is the surface tension of water, r the radius if the interaction area A'_c and factor 4 is created because we consider surface tension to affect both the inner sealing of the system and the outer sealing, which creates to circles of water beneath the coin with very nearly the same radius r, therefore the entire length affected by surface tension is $2 \cdot 2\pi r = 4\pi r$

$$\rho_n = \rho_0 \frac{1}{1+\gamma\left(\Delta T_i + \sum_{i=1}^{n} \Delta T_n\right)}$$
$$\rho_0 = \rho_i \frac{1}{1+\gamma \Delta T_i} \tag{2.8a}$$

Given the initial air density ρ_0, the subsequent values for air density at the n-th lift off are calculated in dependency of the temperature differences required for the next lift-off to occur. This ΔT_n is precisely corresponding to ΔT_{crit}, which itself is changing with time. We use the equation of state presented in equation (2.6) to predict the value of the initial lift-off temperature difference (i.e. the temperature difference required for the first lift-off to occur, i.e. when the absolute temperature T is $T = T_i + \Delta T_1$):

$$\Delta T_n = \frac{(am_c g + 2\pi r\sigma) V_B}{A_c n_n R}$$
$$\Delta T_0 = \frac{(am_c g + 2\pi r\sigma) V_B}{A_c n_0 R} \tag{2.8b}$$

It can be seen that the temperature difference is dependant on the number of air particles in the system. This number can be determined using the following identities:

$$n_n = \frac{\rho_n V_b}{M} \text{ with } n_0 = \frac{\rho_0 V_b}{M} \tag{2.8c}$$

Above, number of particles n is expressed as mass over molar mass. Mass can be expressed as density times volume, which creates a relationship between equations (2.8a) and (2.8c), and couples equations (2.8a), (2.8b) and (2.8c). The last prediction that can be made using the discrete model is to inverse Newton's law of cooling presented in equation (2.1) to predict the time steps between subsequent lift offs:

$$\Delta t_n = \frac{-1}{k} \ln\left(\frac{T_i + \left(\sum_{i=1}^{n} \Delta T_i\right) - T_a}{A}\right)$$
$$\Delta t_0 = \frac{-1}{k} \ln\left(\frac{T_0 - T_A}{A}\right) \tag{2.8d}$$

With this, let us now consider the theoretical aspects of the sound in the task.

2.4 Sound model

At first it may seem improbable to accurately predict the source of the sound of the phenomenon, or its frequency. Keep in mind that the sources of the sound may be the bottle material oscillating, the coin oscillating, or the air in the bottle much like a *Helmholtz resonator*[6]. First consider the bottle material oscillating. In a simplified approach, it can safely be assumed that the bottle neck only resonates and the change in bottle geometry as soon as there is a significant change in curvature of the bottle (which occurs, as soon as the bottle neck ends) will dampen out any sound waves going beyond this point. If only the bottle neck resonates, it can be stated that there are two possible ground modes of resonance: one closed-end ground model and one open-end ground mode. Former has two nodes of the sound waves - one at either end of the bottle neck. The latter has one node at the end where the bottle neck curves into the body of the bottle and an antinode at the opening of the bottle neck. Closed end-resonance is likely not to occur because the bottle can more easily oscillate in the open-end configuration. This is shown in figure 1, where h gives the length of the bottle neck. The length of the bottle neck can be expressed as

$$h = \frac{\lambda_0}{4} = \frac{3\lambda_1}{4} \tag{2.9}$$

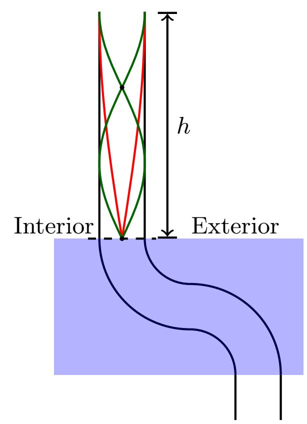


Figure 1: Open-end modes of resonance. The blue area shows the zone in which the curvature of the bottle leads to perturbations of sound waves travelling along the bottle neck

Using the identity of $f = \frac{c}{\lambda}$, where f is frequency, c is speed of sound and λ is wavelength, the following progression can be formulated which describes the upper modes of the open-end resonance in dependency of the ground mode:

$$f_n = (2n+1) f_0 \qquad f_0 = \frac{c}{4h} \tag{2.10}$$

The modes for the coin resonance (both free- and clamped edge) can be found in literature, i.e. in Fletcher and Rossing, 1991. With this, the Helmholtz resonance remains. The Helmholtz resonance frequency is given by

[6] A body of air oscillating at a particular frequency due to pressure discrepancies

$$f_H = \frac{c}{2\pi}\sqrt{\frac{A}{lV}} \qquad c = \sqrt{k\frac{RT}{M}} \tag{2.11}$$

where c is the speed of sound, A is the cross-sectional area of the bottle neck opening, l is its length and V the volume of the entire bottle, k is the heat capacity ratio, R the ideal gas constant, T temperature and M molar mass. It can be concluded, thus, that the speed of sound in air depends highly on temperature. Comparing, for instance, the Helmholtz frequencies at $-10°$C and at $20°$C yields that

$$f_{-10°\mathrm{C}} \approx 0.944 f_{20°\mathrm{C}} \tag{2.12}$$

3 Experimental setup

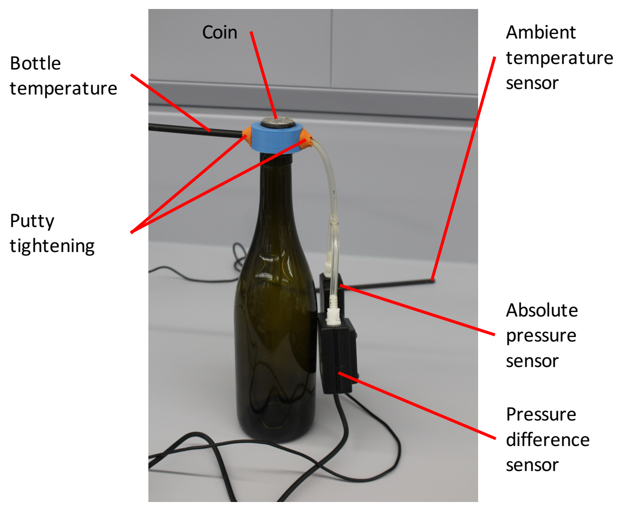


Figure 2: Experimental setup. The tightening ring was latter omitted as it proofed to be redundant in later cap versions.

The experimental setup used constituted itself of several bottles of different materials and surface properties, all of whom influence the rate of change of temperature according to equation (2.5). Furthermore, a set of coins was investigated. Attention was paid to strict adherence to the task statement in that only coins in the sense of coins used for monetary purposes were used. This makes the experiment more difficult because these coins exhibit a rough surface, thereby considerably increasing the chance of a leak occurring in the tightening of the system, thereby destroying the experiment. In order to keep the interaction are between coin and the air in the system constant, a cap construction (visible as the blue piece in figure 2) was 3D-printed. This makes data more comparable. It, however, makes experimenting more challenging as well because there can more easily be a leak in the tightening. As can be gathered from figure 2, ambient temperature, system pressure and system temperature was measured. Ambient pressure was measured once before the onset of the system measurement. During experiment, attention was paid to keeping external factors constant, i.e. not opening windows to change ambient temperature or to create breeze, both of which have an impact on equation (2.5). It is clear, also, that the temperature is not homogeneous in the system due to the parallel flows of heat according to equation (2.5). This was not accounted for, but could theoretically by applying Fourier's law of heat transfer. The temperature sensors are accurate within an uncertainty of $\pm 0.5°$C, and the pressure sensors within an uncertainty of $\pm 2\%$.

4 Data

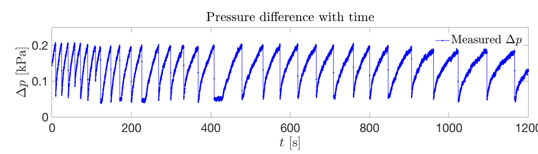
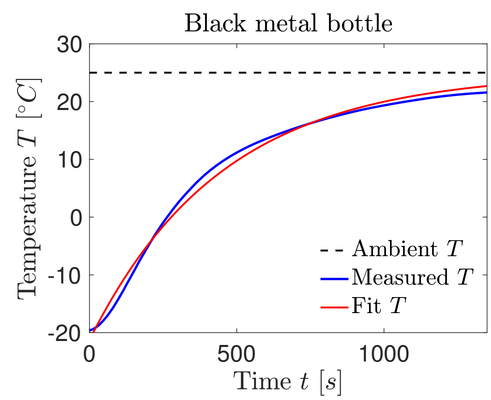
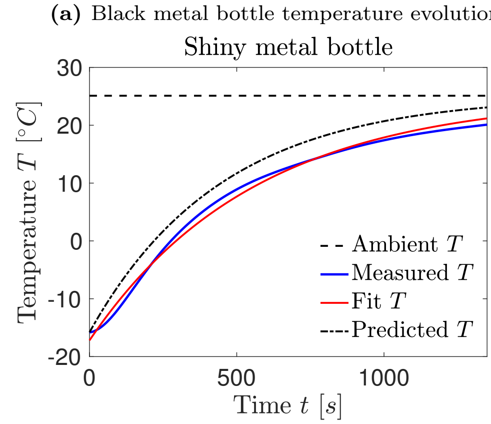


Figure 3: Optimal data. Particularly interesting is a data pattern precisely as suggested by the qualitative model.



(a) Black metal bottle temperature evolution



(b) Shiny metal bottle temperature evolution

Figure 4: Comparison between fitted and model predictions and data. The model values needed for equation (2.5) where taken from literature - the most accurate values that could be found.

In figure 3, it can be seen that the pressure evolution adheres to the qualitative model in that it builds up until it reaches a critical pressure, after which it drops again.

In figure 4, a comparison between the fit and the model becomes obvious in temperature data. It furthermore is interesting that the black bottle shows a faster change in temperature, which is reasonable due to its surface being more alike to a black body radiator than the shiny metal bottle.

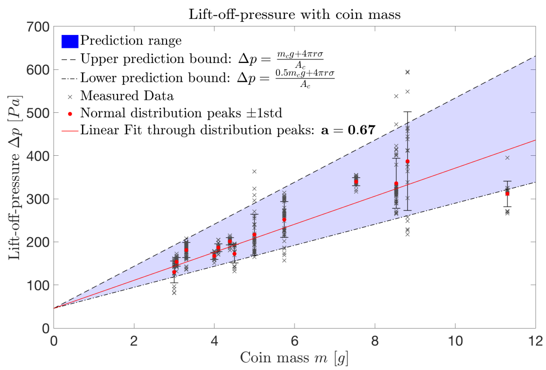


Figure 5: Lift-off pressure with changing coin mass. The black crosses are all the measured data points, the red dots give the peak of the standard distribution curves with the error bars pertaining to the standard deviations of the data points. The red line is the fit through the normal distribution peaks, giving a value of $a = 0.67$. The upper boundary of the (blue) prediction range is derived with $a = 1$ and the lower with $a = 0.5$

Note that figure 5 gives an experimental value of $a = 0.67$. This value is used in further predictions:

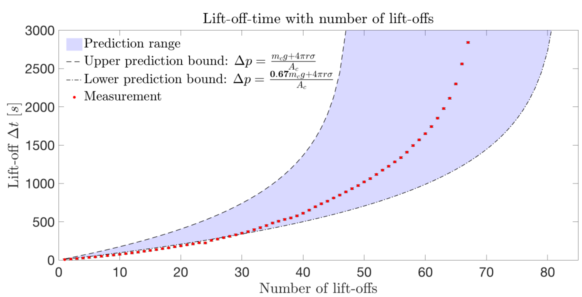


Figure 6: The prediction range is established using $a = 0.67$ established in figure 5. The upper prediction bound is still $a = 1$ because when changing the ratio between coin radius and interaction area radius, it may be that this factor a changes. The bottle used in this experiment was a small glass bottle and the coin used was a Swiss 2-franc-coin.

With this, turn your attention now to the sound model. Figure 7 shows the predictions made from the simplified open-end neck resonance model. In figure 8, a possible explanation of the minor peaks is presented as a result of upper modes of the Helmholtz resonance frequencies of the bottle. It shows the same coin and the same bottle at different temperatures. Note that the black line in the upper plot is at 1848 Hz and the one in the lower at 1727 Hz. The ratio of those two frequencies is $\frac{f_{-10^\circ\mathrm{C}}}{f_{20^\circ\mathrm{C}}} = 0.934$. Comparing this value to the predicted ratio value of the Helmholtz resonances according to equation (2.12), this data is evidence that the theory to explain the minor peaks may be holding, especially given that the deviation between theory and data is less than 1%.

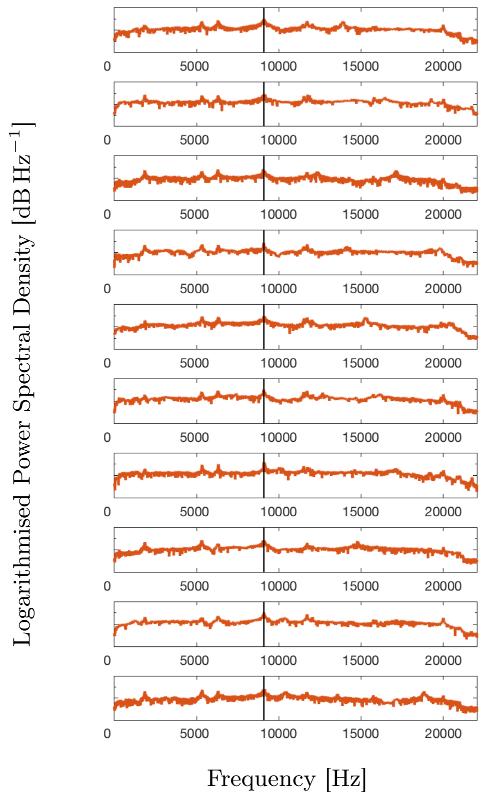


Figure 7: The peaks of the sound spectra superimpose perfectly with the predicted ground mode frequencies of the open-end neck resonance ground mode (black lines in the spectra). Data was taken from ten different coins. From top to bottom, the coins are: 1 Euro, 0.2 AUD, 0.2 CHF, 1 CHF, 2 CHF, 5 CHF, 2 Euro, 0.1 NZD, 0.5 NZD, 0.2 SGPD.

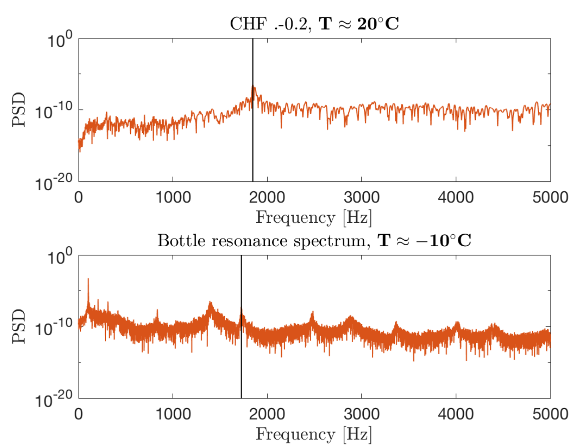


Figure 8: Possible explanations of minor peaks due to temperature-dependant shifts of upper mode-frequencies of Helmholtz resonance. [PSD]=dB Hz^{-1}

5 Interpretation and discussion

The experimental setup used in this experiment overall gave satisfactory results: it was possible to monitor the relevant physical entities relatively easily using the cap construction outlined in section 3 *Experimental setup*. The approach to fitting all the relevant measuring devices into a single cap construction allowed for a high efficiency and degree of organisation. However, there were some drawbacks to the design as well. Firstly and foremostly, the cap construction was prone to air leaks, which would disable accurate measurement. Secondly, it was time-intensive to design, print and set up and lastly, it only proofed to be effective in a certain range of coins, which, however, may have to do with the mechanics of the coin motion as well. This will be discussed at a later stage.

The temperature prediction using Newton's law of cooling gives an outlook over the relevant parameters that impact the temperature evolution with a high degree of certitude. However, the fact that the evolution cannot be modelled accurately using literature values limits the predictive power of the model. While the approach of fitting the temperature curve and then using the fitted value gives a good result, it leaves a bit to be desired in terms of model goodness. Furthermore, Newton's law of cooling only predicts the evolution of temperature in the time-domain, while more fundamental equations of heat flow such as Fourier's law of heat transfer would give an overview over the temperature change in both space- and the time-domain. The latter may have given an indication of how temperature fluctuations impact the pressure fluctuations in the bottle, which ultimately may impose limitations on the factor a. Then again, this would perhaps may not be worth the effort as measuring such physical quantities may provide a great challenge, and on top of it, the approach using Newton's law of cooling gives a good result.

The prediction of the critical lift-off-pressure in dependency of the coin mass uses the equations presented in the theoretical model. Firstly, it needs to be said that all the values are within the prediction boundaries. There are some larger fluctuations off the statistical mean especially in heavier coins, which may have the following explanation: as the pressure increases, the air eventually is forced out of the bottle. However, the pressure needs to compensate both the surface tension force of the water as well as the coin weight. The „way of least resistance" for the air is to overcome the surface tension of the water, rather than lifting the coin. Therefore, not the entire pressure predicted by theory will build up. Likewise, there may be higher pressures if the forces to be overcome are larger than surface tension and weight. Reasons for this may be that the water is not evenly distributed on the cap or surface texture effects of the coins. The statistical analysis to determine factor a leaves a relatively large range of prediction. This is visible in figure 5, but especially in figure 6. In figure 5, there is a visible tendency that standard deviations increase with increasing coin mass in data. The problems outlined above may be the reason for this. In figure 6, the range of prediction becomes very large with increasing number of lift-offs. This is problematic as the coherence of data and theory may not be interpreted unambiguously. Interesting in particular is that until about the 30th lift-off, the prediction that is based on the value of a that was found in the statistical analysis based on figure 5, seems to adequately predict the behaviour. However, there seems to be a deviation as soon as the number of lift-offs becomes greater. The reason for this behaviour is not very clear. With increasing number of lift-offs, the temperature in the system increases gradually towards ambient temperature. Consequently, it will take longer for the subsequent lift-off to occur (which is precisely shown in the model). However, with increasing time interval, it may be that once again the pressure will find the water tightening to offer less resistance than lifting the coin would. As overcoming surface tension forces requires less pressure, less time would be required to lead to a drop in pressure, which was the physical process actually measured to determine the time of a lift-off-event. This explanation, however, does not explain why this phenomenon is not observed in prior lift-offs. The explanation to this should be subject to further research.

Considering the sound model, the approach via the modes of resonance is rather promising. However, there is a number of factors that have a massive impact on the Eigenfrequencies of the bottle, foremostly its geometry. Thus, given a distinct bottle shape where perturbations most likely can occur at a particular zone between bottle neck and bottle body, the approach using the open-end resonance modes may produce good predictions. As soon, however, as the geometry of the bottle is such that the transition zone between neck and body is not easily distinguishable, the simplified approach using the Eigenfrequencies becomes insufficient. This is therefore a limitation to the predictive power of the model. Interesting in particular is that the frequency peaks of the spectra nicely correlate with data. This is strong evidence that the explanation of the main peaks using the approach with the Eigenfrequencies described above holds at least for the bottle using during experiment - one, where the transition zone between neck and body is very clear. No clear statement, however, can be given when considering different bottles, even though their main peaks should also correlate to an Eigenfrequency. Their value is the issue, as it cannot easily be determined in more complex geometries. Note that the model does not consider the coin to oscillate. The reason for this is quite palpable: Oscillations of the coin could occur at the moment the coin comes crashing back onto the bottle neck. As it then generally flatly lies on the latter, and is furthermore in contact with the sealing water, it is assumed that any oscillations of the coin would be damped. Once again, therefore, does the explanation of the major peaks in the frequency spectra hold.

Modelling the minor peaks has proofed fairly difficult. With a high degree of certitude, there are some higher modes of resonance of the bottle and perhaps even of the coin that can be perceived apart from the main peak explained above. Another approach is an explanation using Helmholtz resonance. This approach is justified by the fact that a number of air molecules leave the bottle, thereby possibly creating the pressure difference necessary for a Helmholtz oscillation phenomenon to occur. A part of the theory is strongly reflected in data, namely the really good correlation of the frequency ratios at varying temperature. This being said, it is not clear how exactly these frequencies are to be interpreted: they are assumed to be higher modes of Helmholtz resonance. It is neither clear, however, if they exist, and if yes, how they arise. Regardless of whether or not it is upper modes of Helmholtz resonance that cause some of the minor peaks, the very accurate prediction of the Helmholtz resonance frequency ratio and the frequency ratio of minor peaks in data strongly suggests that at least part of the minor peaks in the spectra are the result of a form of air oscillation. The temperature-dependent differences in the speed of sound of air, therefore, perhaps form the basis of a more holistic approach to explaining the perceived sound.

6 Conclusion and Acknowledgements

The task has been to take a strongly cooled bottle and to put a coin on its neck, and then to explain the subsequent lifting-like motion of the coin and the sound that follows it, as well as to investigate the relevant parameters. Given that a temperature difference of 40 K is considered significant enough, the lifting of the coin was found to be due to pressure dynamics in the system. The sound, on the other hand, cannot be fully explained, even though some promising approaches are presented. The relevant parameters, witch are coin mass, the presence of absence of water tightening and bottle parameters influencing the rate of change of temperature in the system when considering the lift-off, and bottle geometry and material when considering the sound, respectively, have been investigated and modelled. Subject of further research may exemplary be the predict the precise motion of the coin, such as the angle of lift in dependency of coin mass and temperature and other aspects of the precise motion of the coin. Further investigation would also have to be done into the influence of coin surface texture and other factors possibly influencing the required pressure for a lift-off. Very interesting would also be to investigate the proposed limitation to the lift-off-model by the lift-off-pressure discrepancy that could be explained by heavier coins.

As always, research is not done alone but in a team. I have had the honour of presenting the findings above at the IYPT 2018 in Beijing, China, prior to which I have enjoyed tremendous support from the tutors of SYPT: Thank you, Mrs. Emilie Hertig, Mr. Daniel Keller, Mr. Eric Schertenleib. My gratitude furthermore goes to my teammates of the IYPT-2018-team of Switzerland. Lastly, I would like to thank Mr. Reinhard Weiss of Gymnasium Kirschgarten for providing help, support and motivation.

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